

Complex base numeral systems.

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Abstract

In this paper will be introduced large, probably complete family of complex base systems, which are 'proper' - for each point of the space there is a representation which is unique for all but some zero measure set. The condition defining this family is the periodicity - we get periodic covering of the plane by fractals in hexagonal-type structure, what can be used for example in image compression. There will be introduced full methodology of analyzing and using this approach - both for the integer part: periodic lattice and the fractional: attractor of some IFS, for which the convex hull or properties like dimension of the boundary can be found analytically. There will be also shown how to generalize this approach to higher dimensions and found some proper systems in dimension 3.

1 Introduction

Standard numeral systems allows us to represent points from \mathbb{R} in some base (z) as a sequence of digits:

$$x = \sum_i a_i z^i$$

In this paper will be shown how to generalize it to 2 dimensions, using the structure of complex plane $\mathbb{C} = \mathbb{R} + j\mathbb{R}$. It was previously done by Donald Knuth [1], who has introduced imaginary base systems, or in [2] where the $j - 1$ base system was introduced and where was shown that such representations can be used to simplify operations involving complex numbers in today's microprocessors.

In this paper will be presented much larger family of complex base system, which seems to be complete.

We expect from the numeral systems, that the representation function is surjective - each point has at least one representation.
But we rather cannot expect injectivity - for example $0.111111..._2 = 1.000000..._2$ -

there is countable number of points that have two different representations.

In the 2-dimensional case such set won't be countable - it will have Hausdorff's dimension in $[1, 2)$ range. I will assume only, that it's Lebesgue's measure is 0.

Definition 1. We will call the numeral system *proper*, if the representation function is surjective and *pseudoinjective*:

$\exists_{S \subset \mathbb{C}} : \mu_2(S) = 0, \forall_{x \in \mathbb{C} \setminus S} x$ has at most one representation.

Choose some $\mathbb{N} \ni n \geq 2$.

We will use $\bar{n} := \{0, \dots, n-1\}$ digits and some base $z \in \mathbb{C}$.

We have to analyze two completely different sets:

Integral part: $I_z = \{\sum_{i \geq 0} a_i z^i : a_i \in \bar{n}, \exists_N \forall_{i > N} a_i = 0\}$

Fractional part: $F_z = \{\sum_{i < 0} a_i z^i : a_i \in \bar{n}\}$

For the binary system $n = 2$, $z = 2$, $\bar{n} = \{0, 1\}$, $I_2 = \mathbb{Z}$, $F_2 = [0, 1]$.

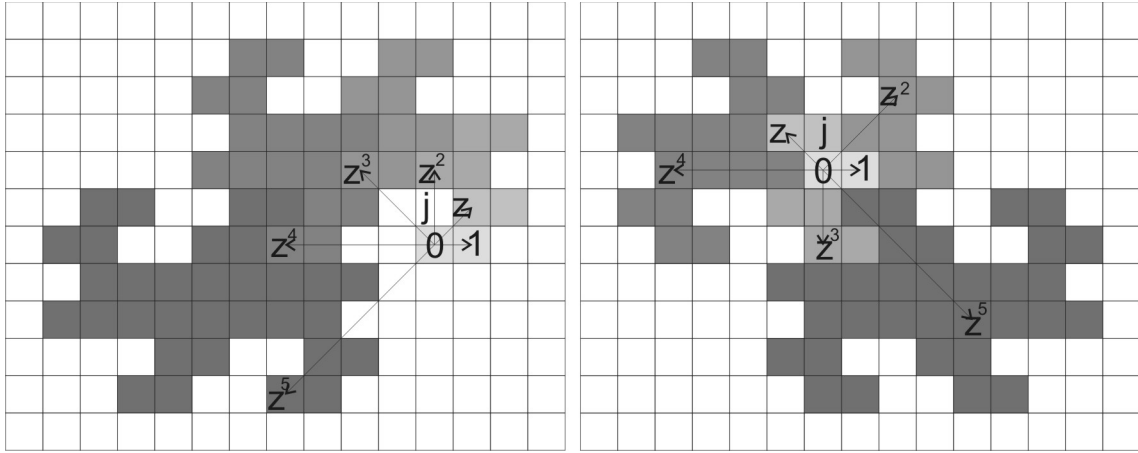


Figure 1: Integer part we can get using 6 youngest digits for $n = 2$ and $z = j + 1$ (left) or $z = j - 1$ (right). It looks like in the first case we will make 'spirals' around 0, but in the second we should cover the whole space .

We will check, that they are fulfilling *selfsimilarity equations*:

$$z \cdot F_z = \bigcup_{i=0..n-1} (F_z + i) \quad (1)$$

$$\bigcup_{i=0..n-1} (zI_z + i) = I_z. \quad (2)$$

For the binary system $2[0, 1] = [0, 1] \cup ([0, 1] + 1)$, $2\mathbb{Z} \cup (2\mathbb{Z} + 1) = \mathbb{Z}$.

We will check that the necessity conditions implies:

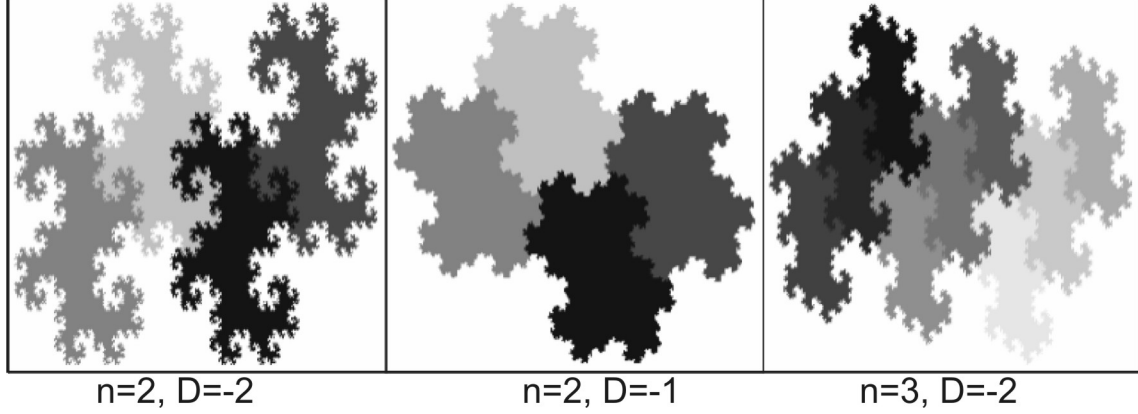


Figure 2: Some fractional parts. The color represents two oldest digits.

Lemma 2. If (n, z) is proper

$$\begin{aligned} \mu_2(F_z) &> 0, \quad \mu_2(F_z \cap (F_z + 1)) = 0, \\ n &= |z|^2. \end{aligned} \tag{3}$$

It's why we will be dropping the upper index of I_z^n, F_z^n .

So F_z is some compact, positive measure, central symmetric([3]) set and when we take copies translated by all points of some discrete set I_z we 'tile' the whole plane, such that tiles should intersect with its neighbors only on the boundaries. Intuition says that it should be *periodic tiling*:

$$z^2 \in \mathbb{Z} + z\mathbb{Z} \tag{4}$$

We will see that it means, that $z = D/2 + j\sqrt{n - (D/2)^2}$ for some $D \in \mathbb{Z} \cap (-2\sqrt{n}, 2\sqrt{n})$.

Hypothesis 3. *All proper (2-dimensional) numeral systems are periodic (4).*

I couldn't prove it. On Fig. 3 are numerically found Hausdorff's dimensions of $F_{\sqrt{2}e^{j\varphi}}$ for $\varphi \in [\pi/2, \pi]$. The tree peaks with dimension two are the periodic cases, while $\alpha \rightarrow \pi$, F_z is becoming a segment - its dimension drops to 1.

We will concentrate on the periodic case in this paper, and show that the picture looks like it seems (like on Fig. 2 or Escher's pictures) - we have slanting two dimensional lattice ($X = \mathbb{Z} + z\mathbb{Z}$) and each of its parallelogram (build of 1 and z) corresponds to exactly one copy of F_z (so $\mu_2(F_z) = |z|\sin(\varphi)$), which is connected, simply connected, $F_z = \text{int}(F_z)$ and it intersects with its 6 neighbors (for $\varphi \neq \pm\pi/2$) on its boundary, which dimension can be calculated analytically.

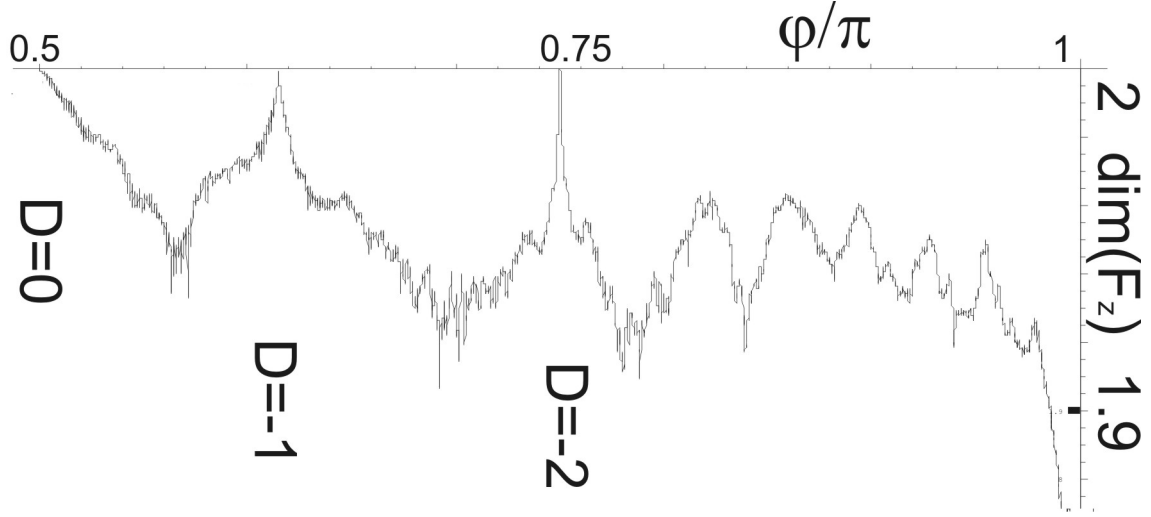


Figure 3: Numerically found Hausdorff's dimension of F_z .

These boundaries (δF_z) are the sets on which points has two or three representations - we can define dense set on which we are losing injectivity - like for the binary system:

$$S = \bigcup_{k \in \mathbb{Z}} z^k (I_z + \delta F_z)$$

In one dimensional case, this set is countable - has dimension equal 0, it's less by 1 than space dimension. The skewness makes that this time, we have a bit worse (see table on page 16).

The surjectiveness condition can be written in a few ways:

$$\mathbb{C} = F_z + I_z = \sum_{i>0} z^i F_z = \overline{\sum_{i<0} z^i I_z} \quad (5)$$

we can check it:

- by checking that $I_z = \mathbb{Z} + z\mathbb{Z}$
- by checking that $0 \in \text{int}(F_z)$

In the next section will be introduced formalism and proven some basic properties: selfsimilarity relations, $|z|^2 = n$ condition, that it's enough to analyze argument of z in one quarter and be characterized the periodic cases.

In the third section we will focus on the integer part. We will show in what cases $I_z = \mathbb{Z} + z\mathbb{Z}$, what is sufficient for system to be proper and that in the rest of cases situation isn't much worse. The main tool will be the reduction - the analogue of shift right with throwing out the youngest digit.

In the forth section we will focus on the fractional part. Firstly there will be

presented results from [3] - a methodology of finding the convex hull of F_z and its properties analytically. In the second - main subsection, will be introduced the methodology of constructing the approximation of the boundary of F_z , which allows to show that the tiling looks 'nice', quickly approximate F_z or to calculate analytically the dimension of the boundary.

In the last section, will be briefly presented a proposition of generalization of this approaches into higher dimensions and shown a family of proper system in dimension 3.

We can use the complex base systems is image compression too - in hexagonal-type structure, we should get better correlations.

The other thing is that calculating the transform of the whole image would cost too much, so the image is usually split into a periodic lattice of squares, which are processed separately. In this approach lossy compression creates recognizable lattice of lines. Splitting into lattice of fractal-like shapes shouldn't have this problem.

To encode inside such shape - $\{\sum_{0 \leq i \leq N} a_i z^i : a_i \in \bar{n}\}$, we can use wavelet transform as it was just usual binary system, for example basic Haar wavelet for $n = 2$ would be: $f_k = 2a_k - 1$ if $\forall_{i>k} a_i = 0$, 0 else.

The boundary of larger integer sets is created from boundaries of smaller ones, so the wavelet functions vanishing on the boundaries should behave well in this case.

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2 Basic definitions and properties

Definition 4. Fix some $2 \leq n \in \mathbb{N}$, $z \in \mathbb{C} : |z| > 1$

Digits $\bar{n} := \{0, 1, \dots, n-1\}$,

Pseudorepresentatnions $P := \{(a_i)_{i \in \mathbb{Z}} : \forall_i a_i \in \mathbb{Z}, \exists_N \forall_{i>N} a_i = 0\}$,

Shift left(right): $\text{shr}(a_i) := (a_{i-1})$, $\text{shl}(a_i) := (a_{i+1})$,

Representations $P^n := \{(a_i) \in P : \forall_i a_i \in \bar{n}\}$,

$P^+ := \{(a_i) \in P : \forall_{i<0} a_i = 0\}$, $P^{n+} := P^+ \cap P^n$,

$P^- := \{(a_i) \in P : \forall_{i \geq 0} a_i = 0\}$, $P^{n-} := P^- \cap P^n$,

Representing function $P \ni (a_i) \rightarrow (a_i)_z := \sum_i a_i z^i$,

Integer part $(I_z \equiv) I_z^n := P_z^{n+}$,

Fractional part $(F_z \equiv) F_z^n = P_z^{n-}$.

We will use standard set arithmetic:

$A + B = \{a + b : a \in A, b \in B\}$, $f(A) = \{f(a) : a \in A\}$.

Observation 5. :

$$(\text{shl}(a_i))_z = z \cdot (a_i)_z, \quad (\text{shr}(a_i))_z = z^{-1} \cdot (a_i)_z$$

$$\text{shl}(P^{n-}) = P^{n-} + (\dots, 0, \bar{n}, 0, \dots), \quad \text{shl}(P^{n+}) + (\dots, 0, \bar{n}, 0, \dots) = P^{n+}$$

So we have the *selfsimilarity equations*:

$$z \cdot F_z^n = F_z^n + \bar{n}, \quad z \cdot I_z^n + \bar{n} = I_z^n.$$

We've just checked (1) and (2) conditions.

The surjectiveness condition means that $P_z^n = \mathbb{C}$

$$P_z^n = \sum_{k>0} z^k F_z = F_z + I_z \quad (6)$$

it's the sum of countable many translated copies of F_z so we must have $\mu_2(F_z) > 0$. Pseudoinjectivity implies that $\forall_{i=1..n-1} \mu_2(F_z \cap (F_z + i)) = 0$ - points in this intersection would have at least two representations - with 0 and with i at zeroth position. Multiplication by z is rotation and rescaling. Rotation and translation doesn't change the Lebegue's measure, and the scaling multiply it by the scaling factor to power dimension - from the selfsimilarity condition we have:

$$n \cdot \mu_2(F_z) = \mu_2 \left(\bigcup_{i=0..n-1} (F_z + i) \right) = \mu_2(z \cdot F_z) = |z|^2 \mu_2(F_z)$$

So we've checked Lemma 2.

Observe that if we assume $|z|^2 = n$, from the derivation above: $\mu_2(F_z) > 0$ automatically implies pseudoinjectivity - we have only to check the surjectiveness.

For the rest of the paper denote:

$$z = \sqrt{n} e^{j\varphi}. \quad (7)$$

When we conjugate z , F_z will be conjugated.

When we take $-z$, we get $(-\bar{n} = \bar{n} - (n-1))$

$$F_{-z} = \sum_{i<0} z^i (-1)^i \bar{n} = \sum_{i<0} z^i \bar{n} - \sum_{\text{odd } i<0} z^i (n-1) = F_z - \frac{(n-1)z^{-1}}{1-z^{-2}} = F_z - \frac{(n-1)z}{z^2-1} :$$

Observation 6.

$$F_{\bar{z}} = \bar{F}_z \quad (8)$$

$$F_{-z} = F_z - \frac{(n-1)z}{z^2-1}. \quad (9)$$

We will now find possible z , fulfilling *periodicity condition* ($z^2 \in \mathbb{Z} + z\mathbb{Z}$) and $|z|^2 = n$.

Thanks of (8), we can restrict to positive imaginary part and eventually take conjugation of everything.

$$z = d + j\sqrt{n-d^2} \quad \text{for some } d \in \mathbb{R}$$

$$z^2 = d^2 - n + d^2 + 2jd\sqrt{n-d^2}$$

Comparing imaginary parts, we have $D := 2d \in \mathbb{Z}$, we finally have:

Observation 7. *All periodic cases are:*

$$z = D/2 + j\sqrt{n - (D/2)^2} \quad (10)$$

$$z^2 = Dz - n \quad (11)$$

for some $2 \leq n \in \mathbb{N}$, $D \in \mathbb{Z} \cap (-2\sqrt{n}, 2\sqrt{n})$.

In the next section we will prove:

Theorem 8. *Complex system fulfilling the periodicity condition is proper iff $D \leq 1$.*

3 The integral part

In this part we will check whether $I_z = X$, where we will denote

$$X := \mathbb{Z} + z\mathbb{Z}.$$

Because $I_z + F_z = \overline{\bigcup_{k < 0} z^k I_z}$, we'll have checked the Theorem 8 then.

3.1 The general behavior

Fix some n, D, z as in Observation 7.

We have $n \geq 2$, $D^2 < 4n$ - it's easy to check:

$$D \leq n, \quad n = D \Rightarrow n \in \{2, 3\} \quad (12)$$

The equation $z^2 = Dz - n$ can be written as

$$(n, -D, 1)_z = 0 \quad (13)$$

where we use the natural identification:

$$\mathbb{Z}^k \ni (b_0, \dots, b_{k-1}) \equiv (a_i) : \forall_{i < 0 \vee i \geq k} a_i = 0, \quad \forall_{0 \leq i < k} a_i = b_i.$$

Now take some $\mathbb{C} \ni x := (b_0, \dots, b_k)_z$, where $\forall_i b_i \in \mathbb{Z}$.

If we add to this pseudorepresentation of x any multiplicity of $(n, -D, 1)$ or some of its shift, it will still represent x . If we subtract $\lfloor b_0/n \rfloor (n, -D, 1)$, we will make the zeroth digit to be in \bar{n} .

It suggest an algorithm to change a pseudorepresentation into representation - use above construction on succeeding digits to bring them into \bar{n} .

The question is: will it finally stop?

Example 9. We can use $(n, -D, 1)_z = 0$ in many ways.

$$n = 3, D = 3 \Rightarrow z = \frac{3}{2} + j\frac{\sqrt{3}}{2}, (3, -3, 1)_z = 0$$

We can change some pseudorepresentation into representation:

$$(7, -8, 7, -2) \rightarrow (1, -2, 5, -2) \rightarrow (1, 1, 2, -1) \rightarrow \dots \rightarrow (1, 1, 0, \dots, 0, 2, -1)$$

Find the representation of some $x \in X$:

$$x = 10z - 12 = (-12, 10)_z = (0, -2, 4)_z = (0, 1, 1, 1)_z = z + z^2 + z^3$$

Or find the position on X for some representation:

$$z^4 = (0, 0, 0, 0, 1)_z = (0, 0, -3, 3)_z = (0, -9, 6)_z = (-18, 9)_z = 9z - 18,$$

generally: $z^k = \begin{pmatrix} z & 1 \end{pmatrix} \begin{pmatrix} D & 1 \\ -n & 0 \end{pmatrix}^k \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

Digression We can use arithmetic in such bases similar as usual - for example adding would be adding two representations position by position - we get a pseudorepresentation of the sum. We have to change it into representation. In the binary system we would use $(2, -1)_2 = 0$, this time $(n, -D, 1)_z = 0$.

We can create parallel algorithm for it - in the binary system we divide the representation into two parts and sum them parallelly, for every possible carried digit (0 or 1). The difference in this case, is that we have more 'carried cases': for addition from 8 (for $D = 1$) to 21 (for $n = 3, D = 3$).

$x \in \mathbb{Z} + z\mathbb{Z} = x_0 + zx_1$ can be 'psuedorepresented' as $x = (x_0, x_1)_z$, so

Lemma 10. For any $x \in X$, there exists its *reduction* $r(x) := y \in X$, such that:

$$x - yz \in \bar{n}.$$

We can think about the reduction as throwing out the youngest digit ($\in \bar{n}$).

We will now check if r have a nonzero fixed point(s).

Assume that $r(x) = x$ for some $x = a + zb$:

$$r((a, b, 0)_z) = r((a - kn, b + kD, -k)_z) = (b + kD, -k)_z$$

where $k = \lfloor a/n \rfloor$.

$a + bz = (b + kD) - kz$, so $b = -k$, a and b has different signs,

$a = b + kD = b - bD = b(1 - D)$ - we must have $D > 1$,

$b = -\lfloor b(1 - D)/n \rfloor = \lceil b\frac{D-1}{n} \rceil$, but $0 < \frac{D-1}{n} < 1$ (12) so $b > 0$

for $b = 1$ we have $(1 - D, 1)$ for any $D \geq 2$,
 for $b = 2$, we would need $2(D - 1) > n$ - it can happen only for $n = 3$, $D = 3$ - it's very special case we would focus later on.
 Because of (12) it's the largest possible b .

We have that there are 1-3 fixed points of r . We would like to show that from all points of X we would get in finite number of reductions to one of these points.

Lemma 11. $|x| > \sqrt{n} + 1 \Rightarrow |r(x)| < |x|$

Proof: $|r(x)| = \left| \frac{x-a}{z} \right| \leq \frac{|x|+(n-1)}{|z|} = \frac{|x|+(\sqrt{n}+1)(\sqrt{n}-1)}{\sqrt{n}} < \frac{|x|+|x|(\sqrt{n}-1)}{\sqrt{n}} = |x|. \quad \square$

Because we are in the discrete lattice X , starting from any point $x \in X$ after a finite number of reductions, we will get into the ball $B := \{x \in X : |x| < \sqrt{n} + 1\}$. It's finite set, all fixed points are in it - if we would show that there are no cycles in it, the reduction process will always stabilize.

Definition 12. *Attractor* of fixed point s , is $A_s := \{x : \exists_i r^i(x) = s\}$.

B is finite, so for a finite number of cases we can do it by checking that in fact all points of B will be in some attractor. I've done it for $n = 2, 3$ (Fig. 4).

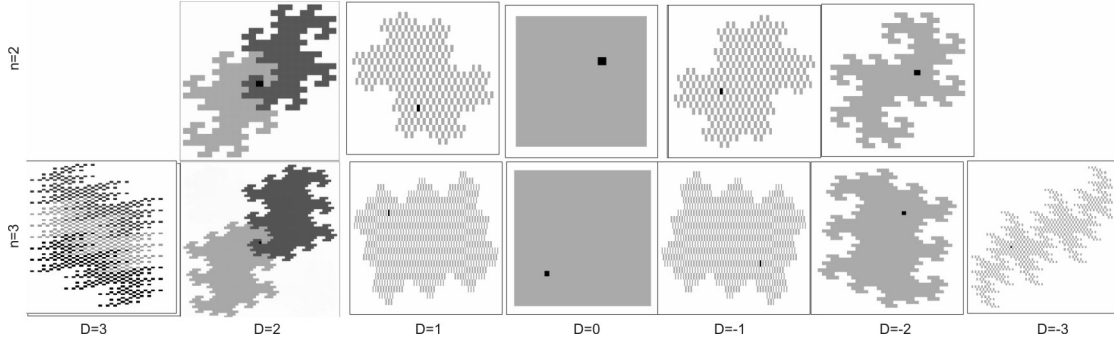


Figure 4: All cases for $n = 2, 3$. Different colors represents different attractors. Black dot is the center of coordinates.

For $n \geq 4$, $\sqrt{n} + 1 \leq n - 1$, so $\bar{n} + z\bar{n}$, which is in the attractor of 0, covers $\{x \in B : \arg(x) \in [0, \varphi]\}$ (see Fig. 5).

$\varphi \in (0, \pi) : \text{if } \arg(x) \in [\pi, 2\pi), \arg(r(x)) < \arg(x)$.

So finally this cycle could be only in $\{x \in B : \arg(x) \in (\varphi, \pi)\}$.

If $\varphi > \pi/2$, ($D < 0$), after one reduction we get into the $\bar{n} + z\bar{n}$, else after 2 reductions we get into the second stationary point.

We've just proven Theorem 8. \square

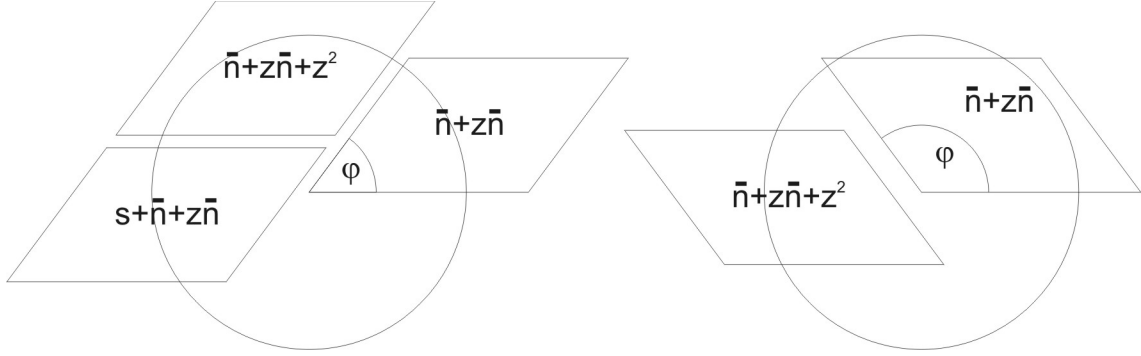


Figure 5: Constructing a cycle.

3.2 Special cases

We will now concentrate on the $D \geq 2$ cases: when $I_z \neq X$.

In this case we can split X into two or three disjoint subsets (attractors of fixed points) - we can still use z as a numeral system, but this time we have to remember the number of fixed point too (or of cycle for some cases in higher dimensions). For each fixed point $s = r(s)$, we can split the attractor into subsets (A_k) of points which have the same number of reduction (k) to get to the fixed point (we need k digits to encode it):

$$zA_s^k + \bar{n} = A_s^{k+1}, \quad r(A_s^{k+1}) = r(A_s)$$

So because $A_s^0 = \{s\}$,

$$A_s^k := sz^k + \sum_{0 \leq i < k} z^i \bar{n} \quad (= sz^k + A_0^k)$$

Using $A_s = \sum_{k \geq 0} A_s^k$ $A_s^0 \subset A_s^1 \subset \dots \subset A_s^k \subset \dots$, we have

$$zA_s + \bar{n} = A_s \tag{14}$$

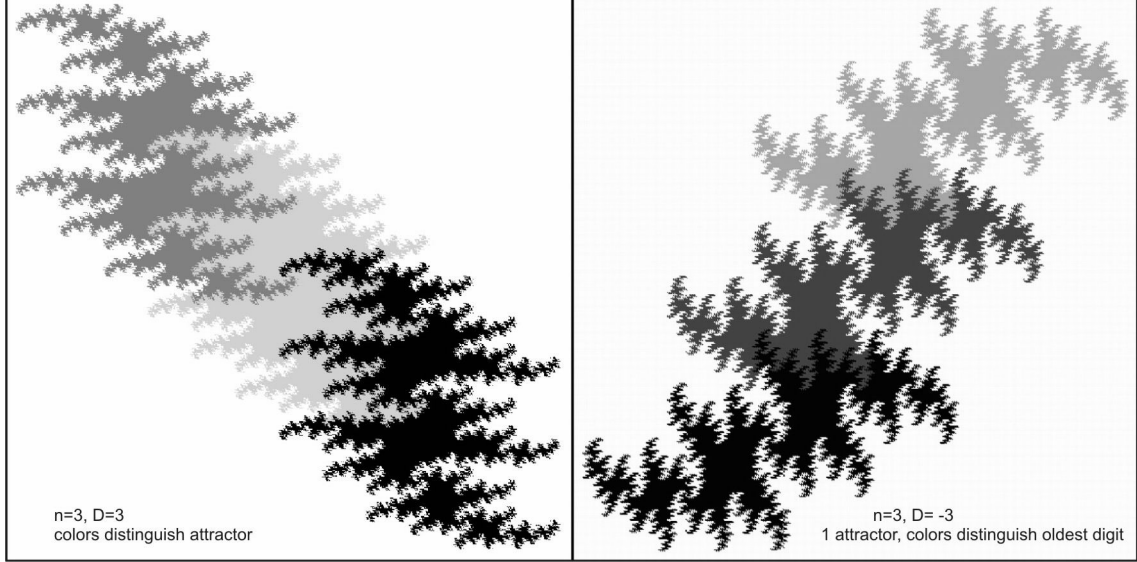
it's the same similarity equation as for I_z (2).

There can be some nice symmetries here: check for which x_0 , $x_0 - I_z$ fulfills (2)

$$z(x_0 - I_z) + \bar{n} = z(x_0 - I_z) + n - 1 - \bar{n} = zx_0 + n - 1 - (zI_z + \bar{n}) = zx_0 + n - 1 - I_z$$

If we want the right side to be equal $x_0 - I_z$, we must have $zx_0 + n - 1 = x_0$

$$x_0 = \frac{n-1}{1-z} \tag{15}$$

Figure 6: The difference for $D = \pm 3$.

For $D = 2$, we have $x_0 = \frac{n-1}{1-D-j\sqrt{n-1}} = j\sqrt{n-1} = z-1 \in X$,

For $n = 3$, $D = 3$, $x_0 = 2z-4$ we can say, that A_{z-2} separates this two copies.

Unfortunately for the rest cases $x_0 \notin X$ - we can see it using

$$A_0^k = \sum_{i=0..k-1} z^i \bar{n} = \sum_{i=0..k-1} z^i (n-1-\bar{n}) = \sum_{i=0..k-1} z^i (n-1) - A_0^k = (n-1) \frac{z^k - 1}{z - 1} - A_0^k$$

so $A_s^k = sz^k + (n-1) \frac{z^k-1}{z-1} - A_0^k$, but this point moves with k in the other cases.

Finally

Theorem 13. We can have the following periodic cases:

- For $D \leq 1$, $X = A_0$,
- for $D = 2$, $X = A_0 \cup A_{z-D+1} = I_z \cup (z-1-I_z)$,
- for $n = 3$, $D = 3$, $X = A_0 \cup A_{z-2} \cup A_{2z-4} = I_z \cup (2z-4-I_z) \cup A_{z-2}$
- for the rest of cases $X = A_0 \cup A_{z-D+1}$.

4 The fractional part

In this section will be shown the methodology to analyze F_z - attractors of simple iterated functional systems.

4.1 The convex hull

I will shortly present the results from [3], which allows to find analytically the convex hull of such simple fractals.

The idea is to define some function which can be easily rewritten in the selfsimilarity form - we get some functional equations, which can be solved analytically or approximated.

The function we need in this case, is the width function, which gives for every direction the width in that direction.

The other functions which can be threaded in such way are for example:

$$f(x) = \mu(F_z \cap (F_z + x)) \text{ (some Hausdorff's measure now),}$$

$$f_p(x) = \sum_{y \in I_z} (x - y)^{-p} \text{ (for example } p = 2)$$

$$\tilde{f}(x) = \sum_{y \in I_z} e^{jx \cdot y} \text{ (or } = \int_{F_z} e^{jx \cdot y} dy \text{ (this time '}' means scalar multiplication).}$$

In our cases, the equations for the width function can be solved analytically. They defines the convex hull of a set as the intersection of all halfplanes:

$$\text{conv}(F_z - x_0) = \bigcap_{\alpha \in [0, 2\pi)} \{x : \Re(e^{-j\alpha} x) \leq h(\alpha)\}$$

$$h(\alpha) = \frac{n-1}{2} \sum_{j>0} r^{-j} |\cos(\alpha + j\varphi)|$$

$$x_0 = \frac{1}{2} \frac{n-1}{z-1}$$

where x_0 is its center of symmetry, $h(\alpha)$ gives the position of bounding line in α direction. There is also shown how to construct analytically this convex hull from triangles, find the length of its boundary $(2\sqrt{n} + 2)$ or its area $((n-1) \sum_{i>0} |\sin(i\varphi)| \sqrt{n}^{-i})$.

4.2 The construction of the boundary

In this subsection, will be shown the methodology of using succeeding approximations of F_z :

$$F_z = \frac{\sum_{i=1, \dots, k} z^k \bar{n} + F_z}{z^k} = \bigcup_{k>0} \frac{\sum_{i=1, \dots, k} z^k \bar{n}}{z^k}$$

the first form is true for any $k \geq 0$ - we can think about F_z as a discrete net of smaller copies of F_z .

The second: sum of growing family of such nets, shows that taking the limit $k \rightarrow \infty$, we can 'forget' about *tiles* (copies of F_z) - approximations restores the whole set.

We will introduce discrete versions of some topological properties, like path,

boundary, connectiveness, show how to make the step to the next approximation and finally that in the limit, they really corresponds to their continuous equivalent.

We will finally prove:

Theorem 14. *For periodic cases ($z^2 = Dz - n$, $D \in 2(-\sqrt{n}, \sqrt{n})$), we have:*

- *if $D \neq 0$, we can split (intersecting in at most one point) δF_z into six connected components - common parts with its 'neighboring tiles',*
- *F_z - connected, simply connected,*
- *$\overline{\text{int}(F_z)} = F_z$.*

The purely imaginary case: $z = j\sqrt{n}$ ($D = 0$) can be easily solved [1] - even digits corresponds real part, odd: imaginary, F_z is just a rectangle. In this case $F_z + I_z$ behave not like for the others: hexagonal lattice, but it's just a rectangular lattice - we will omit this case in this subsection.

Look on (9) - changing the sign of D will only transpose F_z - in this subsection we can restrict to

$$D < 0. \quad (16)$$

For approximations we can work with $X = \mathbb{Z} + z\mathbb{Z}$ space:

Definition 15. For any $A \subset X$ define:

Magnification of A is $M(A) = Mz + \bar{n} = \bigcup_{i=0, \dots, n-1} (Mz + i)$.

k-th approximation of A is $M^k(A)/z^k$.

We can think about it that we see only points from discrete lattice (X) and we use magnifying glass, which allows us to increase magnification $|z|$ times in one step (plus rotation).

After such step, every point(tile) occurs to be n points(tiles).

To have the hexagonal behavior, we would need to describe neighbors (Fig. 7). I will just give positions and in the next lemma will be shown how to check them.

Definition 16.

Neighbors index set: $R = \mathbb{Z}_6$ (\mathbb{Z} modulo 6).

r-th neighbor of $x \in X$ (where $r \in R$) is $N_r(x) := x + N_r$, where:

$N_0 := 0, N_1 := -z + D + 1, N_2 := -z + D, N_3 := -1, N_4 := z - D - 1, N_5 := z - D$.

The *r-th meeting place* is $C_r(x) := xz + C_r$, where

$C_0 := n + d, C_1 := n - 1, C_2 := n - 1, C_3 := -D - 1, C_4 := 0, C_5 := 0$.

The *r-th meeting point* is $P_r(x) := x + P_r$, where

$P_0 := 1 + \frac{n}{z-1}, P_1 := 1 + \frac{n-D}{z-1}, P_2 := \frac{n}{z-1}, P_3 := \frac{1}{z-1} - 1, P_4 := \frac{D-1}{z-1} - 1, P_5 := \frac{1}{z-1}$.

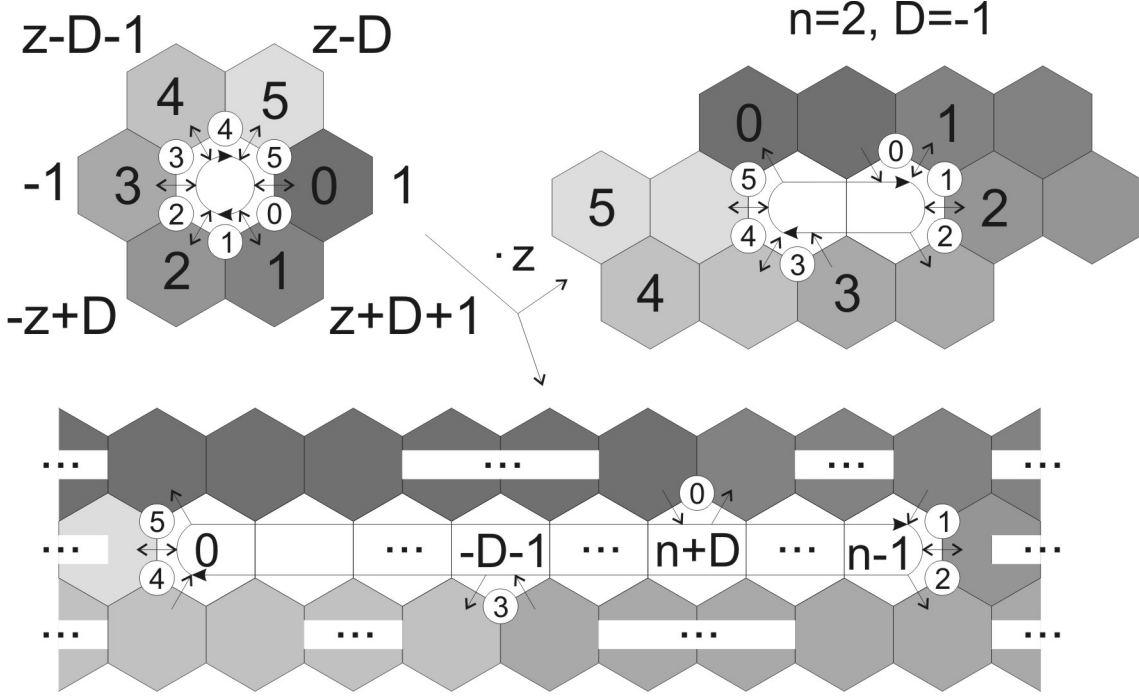


Figure 7: The construction of magnification.

N are the positions of centers of neighboring tiles - we order them clockwise. After one step of magnification, the center tile is split into n tiles, C_r tells in which of them is the common point with the magnifications of r -th and $r + 1$ -th neighbor (dots with numbers on the picture).

We can easily find the real point of this meeting - do the magnification:

$$zP_r = P_r + \text{the position of tile for which } zP_r \text{ is } r\text{-th place of meeting}$$

for example $zP_0 = P_0 + (n + D) + (z - D - 1)$.

We will now check properties for making a step of magnification.

Lemma 17. *For $x \in X$, $r \in R$, we have that $C_r(x) \in M(x)$ is the neighbor of some elements from $M(N_r(x))$ and $M(N_{r+1}(x))$.*

Proof: We will check it for $x = 0$ - we can translate everything to get the thesis. It's easy to check the table below ($z^2 = Dz - n$):

r	0	1	2	3	4	5
N_r	1	$-z + D + 1$	$-z + D$	-1	$z - D - 1$	$z - D$
zN_r	z	$n + z$	n	$-z$	$-n - z$	$-n$
C_r	$n + D$	$n - 1$	$n - 1$	$-D - 1$	0	0
a	$n - 1$	$-D - 1$	0	0	$n + D$	0
r_a	4	5	0	1	2	3
b	0	0	$n + D$	$n - 1$	0	$-D - 1$
r_b	5	0	1	2	3	4
P_r	$1 + \frac{n}{z-1}$	$1 + \frac{n-D}{z-1}$	$\frac{n}{z-1}$	$\frac{1}{z-1} - 1$	$\frac{D-1}{z-1} - 1$	$-\frac{1}{z-1}$

such that $N_{r_a} = zN_r + a$, $N_{r_b} = zN_{r+1} + b$,

for example for $r = 0$: $N_{r_a}(C_r) = n + D + N_4 = z + (n - 1) = zN_r + a$ \square .

Definition 18.

The *neighborhood* of $x \in X$ is $N(x) := \{N_r(x) : r \in R\}$.

The *edge* of $A \subset X$ is $(\bigcup_{x \in A} N(x)) \setminus A$.

A sequence $(x_i)_{0 \leq i < k}$ ($x_i \in B \subset x$) will be called a *length k chain in B* if succeeding elements are neighbors.

Chain is called *closed*, when additionally x_0 and x_{k-1} are neighbors.

Subchain of some closed, length k chain if is created from some of its succeeding elements (0 is after $k - 1$).

A set $A \subset X$ is called *connected*, if any two of its points can be connected by a chain in this set.

A set $A \subset X$ is called *simply connected*, if $X \setminus A$ is connected.

The range of indexes: $[a, b]$, where $a, b \in R$, is called a step 1 arithmetic sequence in R from a to b (eg. $[4, 2] = (4, 5, 0, 1, 2)$).

A set $B \subset X$ *lies on the left of* sequence $(x_i)_{0 \leq i < k}$, when for each $1 \leq i \leq k - 2$: if x_{i-1} and x_{i+1} are correspondingly a and b neighbor of x_i , then

$$\{x_i + N_k : k \in [a + 1, b - 1]\} \subset B.$$

A set $B \subset X$ *lies on the left of* closed chain, if this property is fulfilled for each of its subchains.

We need lying on the left to make closed chain around some set oriented (counterclockwise).

The approximation of the boundary of F_z can be started from length 6 closed chain - neighbors of 0 in counterclockwise order - $\{0\}$ lies on the left of this chain.

Now we have to make a general construction of making the step to the next magnification.

Lemma 19. Let $B \subset X$ lies on the left of some (closed) chain $(x_i)_{0 \leq i < k}$.

Then there exists (closed) chain $(y_i)_{0 \leq i < l}$, such that $y_0 \in M(x_0)$, $y_{l-1} \in M(x_{k-1})$ and that $P(B)$ lies on the left of it.

Proof: Firstly we will find (y_i) for length tree chain: $(N_a, 0, N_b)$ and set $\{N_{a+1}, \dots, N_{b-1}\}$.

Define $y_0 \in M(N_a)$ to be neighbor of C_a , $y_{l-1} \in M(N_b)$ to be neighbor of C_b .

The rest of the chain lies in $M(0) = \bar{n}$, such that the set lies on the left - it's chosen unambiguously - look on Fig. 7 : using some arrow we enter \bar{n} in C_a , go along clockwise cycle inside, to exit turning left in C_b .

For example (denote $(N_a, 0, N_b)$ as a_b): $0_4 \rightarrow 4_0(3_0)^{-D-2}3_3(0_3)^{n+D-1}0_2$.

In the general situation, we are taking succeeding x_i and use the above construction for $(x_{i-1} - x_i, 0, x_{i+1} - x_i)$ (remembering to remove duplicates on the ends). \square

Now starting from C_0 - closed, counterclockwise chain around $\{0\}$, call C_i - the i -th use of above lemma to C_0 .

It's the edge of $M^i(\{0\})$, which lies on the left of C_i .

When we take approximations (divide this situation by z^i), we see that we are getting closer to F_z and its (oriented) boundary.

$$C = \{x \in \mathbb{C} : \exists_{x_i} x_i \rightarrow x, x_i \in C_i/z^i\}$$

Proof of Theorem 14 : We have only to check:

- C is closed - just take diagonal sequence.
- C is connected - the distance between succeeding points of chain C_i/z^i goes to zero.
- $\mathbb{C} \setminus C$ has exactly two connected components - it's true for every approximation, using the previous point we have it in the limit.
- The interior component with C is F_z , $C = \delta F_z - M^i(\{0\})/z^i$ are approximations of F_z and its distance to C_i/z^i is going to 0.
- In each step we can divide C_i/z^i into six subchains - lying in approximations of succeeding neighbors $(M^i(N_a)/z^i)$ - we can split C into 6 subsets this way.
- $\overline{\text{int}(F_z)} = F_z$ - from the Bair's theorem: if $\text{int}(z) = \emptyset$, then the sum of countable number of them couldn't give the whole space.
Because $\forall_k F_z = (M^k(\{0\}) + F_z)/z^k$, we can cover F_z with its smaller copies having arbitrary small diameter - some of them will lie in this nonempty interior. After rescaling back, we've got the thesis. \square

Using the above construction of succeeding approximations of δF_z , we can for example calculate its Hausdorff's dimension.

Namely using Lemma 19, we can change two edges: $(N_a, 0, N_b)$ into a sequence of them in the next magnification.

If for each edge we distinguish between the direction of the next edge - remember

the first of this two edges as a_b , we can assign to it a sequence of such pairs in the next magnification (like in the example, but without the last 0_2).

If we are interested in the number of edges only (instead of above grammar), we can write this iteration in the 36×36 (or 18×18 if we use symmetry) matrix form. It's normalized dominant eigenvector tells us the asymptotic probability distribution of edges, corresponding eigenvalue(λ) tells that asymptotically, for the next approximation we need λ times more edges, but the length of them is \sqrt{n} times smaller, so the boundary Hausdorff's dimension is:

$$H = \frac{\log(\lambda)}{\log(\sqrt{n})}$$

Here are found values for $n \leq 9$:

D/n	0	1	2	3	4	5
2	1	1.210760533	1.523627086			
3	1	1.162039854	1.376841713	1.657559542		
4	1	1.134761994	1.303052340	1.508664987		
5	1	1.116924317	1.257583258	1.422944863	1.608726378	
6	1	1.104171451	1.226294386	1.366294523	1.520716574	
7	1	1.094508825	1.203216411	1.325629733	1.458928649	1.598134771
8	1	1.086882303	1.185363533	1.294784127	1.412801706	1.535582008
9	1	1.080677473	1.171064018	1.270444950	1.376841713	1.487192945

In our cases, δF_z in fact - defines F_z , and it's dimension is smaller.

So we can use just the boundary - operating on it (drawing for example) is much faster.

5 Higher dimensions

In this informal section there will be shortly introduce a suggestion of higher dimensions generalization, focusing on the dimension 3.

Once more I will do it only for the periodic case.

I haven't even checked the hypothesis numerically in this case.

We can think about z^k from the previous sections as

$$z^k := Z^k \mathbf{1} \quad Z^k \mathbf{1} \cdot Z^l \mathbf{1} := (z^k z^l = z^{k+l} =) Z^{k+l} \mathbf{1} \quad (17)$$

where $\mathbf{1} = (1, 0)^T$, $Z = r \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}$.

We can use this definition (17) in higher dimension Euclidean space.

We will focus on \mathbb{R}^3 .

Choose some unit vector ($\mathbf{1}$), $r \in \mathbb{R} : |r| > 1$ and some orthogonal matrix $O \in \mathbb{R}^{3 \times 3}$.

Define

$$Z := rO$$

similarly as for complex numbers.

We can diagonalize O - it is a rotation (φ radians) around some vector (u) and eventually the symmetry, but we can put it in the sign of r .

We want $1, z, z^2$ to generate whole space, so the angle between $\mathbf{1}$ and u (denote it ψ) cannot be the multiplicity of $\pi/2$.

So we can define orthonormal basis of our space: $e_1 := \mathbf{1}$, e_2 is orthogonal to e_1 and u lies on the space generated by them, e_3 is orthogonal to e_1 and e_2 .

Define the another basis $\{u', v, w\}$:

$u' = \cos(\psi) \cdot (\cos(\psi), \sin(\psi), 0)$ - rescaled u ,

$v = \sin(\psi) \cdot (\sin(\psi), -\cos(\psi), 0)$ - orthogonal to u ,

$w = e_3$.

Now we can join v and w and think about our space as $\mathbb{R} \times \mathbb{C}$:

$\mathbf{1} = (1, 1)$, we have multiplication and addition by coordinates,

$$z^k = (r^k, r^k e^{jk\varphi}) \quad (18)$$

We can define the analog of the conjugacy: $\{e_1, e_2\}$ plane symmetry $\left(\overline{(x, y)} := (x, \bar{y})\right)$.

Analogically to (3), we have the condition:

$$r = \sqrt[N]{n}$$

in N dimensional space.

To assure the periodicity in dimension 3, we would need ($|r|^3 = n$):

$$z^3 = -Az^2 - Bz - C\mathbf{1} \quad (19)$$

for some $A, B, C \in \mathbb{Z}$.

conjugate this equation and multiply by z^3 :

$$r^6 \mathbf{1} = -Ar^4 z - Br^2 z^2 - Cz^3$$

comparing both equations, we get (the sign of C is hidden in r):

$$C = r^6/C, \quad A = Br^2/C, \quad B = Ar^4/C$$

$$C = -r^3, \quad B = -Ar$$

We see that r have to be integer. We've found the analog of (13):

$$(-r^3, -Ar, A, 1)_z = 0 \quad (20)$$

We now have to find some $z = (R, Re^{j\varphi})$ fulfilling this equation.
The real part gives:

$$0 = -r^3 - ArR + AR^2 + R^3$$

$R \in \mathbb{R}$, so $R = r$.

The complex part gives now:

$$\begin{aligned} 0 &= -r^3 - Ar^2 e^{j\varphi} + Ar^2 e^{2j\varphi} + r^3 e^{3j\varphi} = r^2 (r(e^{3j\varphi} - 1) + Ae^{j\varphi}(e^{j\varphi} - 1)) \\ -A/r &= e^{-j\varphi}(e^{2j\varphi} + e^{j\varphi} + 1) = e^{j\varphi} + 1 + e^{-j\varphi} = 1 + 2\cos(\varphi) \\ \varphi &= \arccos\left(-\frac{1}{2}(1 + A/r)\right) \quad (\Rightarrow A \in \{-3r + 1, \dots, r - 1\}) \end{aligned} \quad (21)$$

The sign of φ is only the matter of conjugation.

Using (20) we can make sequence of reductions like in 3.1.
By analogy to Lemma 11, we have

$$|x| > \frac{n-1}{\sqrt[n]{n}-1} \Rightarrow |r(x)| < |x| \quad (22)$$

we will finally get into this ball, and finally get to a fixed point or a cycle.

We can check that $(0, 0, 0)$ is the only fixed point for $A \geq 0$.

But in this case, for $r > 0$, we get a length two cycle:

$$(2A - 1, -A, -1) \rightarrow (-A, -1, 0) \rightarrow \dots$$

I've checked, that in the second case:

$$m \geq 2, r = -m, n = m^3, A \in \{0, \dots, 3m - 1\}, (n, mA, A, 1)_z = 0 \quad (23)$$

for a few first m everything is fine - we can generate the whole ball starting from $(0, 0, 0)$ - we have proper numeral systems.

We can generalize the methods for the fractional part in this cases too.
The width function in introduced coordinates have two arguments, say $\alpha \in [-\pi/2, \pi/2]$, $\beta \in [0, 2\pi)$ which corresponds to $(\cos(\alpha), \sin(\alpha)e^{j\beta})$.
Now using $z^2 F_z = F_z + \bar{n}(1, 1) + \bar{n}(r, re^{j\varphi})$, we have equations for h , in which α is fixed - we can find solutions as a infinite sum for a fixed α , like in [3].
They don't correspond to one plane (cross section) this time - to find u coordinate, we need to use $dh/d\alpha$.

We can use the method to construct the boundary of F_z .
This time every tile has 14 neighbors:

$$N(0) = \pm\{1, A + z, A - 1 + z, Am + Az + z^2, Am - 1 + Az + z^2, \\ , Am - A + (A - 1)z + z^2, Am - A + 1 + (A - 1)z + z^2\}$$

Now the simplest structure we can separately magnify, analogically to $(N_a, 0, N_b)$, is 0 with closed chain made of its neighbors, which splits $N(0)$ into two connected subsets, which can be distinguished using chain orientation (one of them can be empty).

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